Traces on Noncommutative Homogeneous Spaces

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The noncommutative Heisenberg manifolds constructed by Rieffel in [R2] have turned out to provide interesting examples of C^*-algebras which are similar, but not isomorphic to irrational rotation algebras as shown by Abadie in [A2]. It was shown in [LR2] that these algebras are special cases of a more general construction giving deformations of $C(G/\Gamma)$ for a compact homogeneous space G/Γ. The C^*-algebras obtained were denoted $C^*_r(\widehat{G}/\Gamma,\rho)$ and their ideal structure was determined in [LR2], in this follow-up we shall describe the algebras more closely: (1) The center $\mathcal{Z}C^*_r(\widehat{G}/\Gamma,\rho)$ is isomorphic to $C(G/G^0_\rho)$ for a certain subgroup G^0_ρ of G, and (2) there is a conditional expectation $E^0 : C^*_r(\widehat{G}/\Gamma,\rho) \mapsto C(G/G^0_\rho)$ and therefore a 1-1 correspondence between normalised traces on $C^*_r(\widehat{G}/\Gamma,\rho)$ and probability measures on G/G^0_ρ. This is used to show that $C^*_r(\widehat{G}/\Gamma,\rho)$ also can be represented over $L^2(G/\Gamma)$ just as in the nondeformed case.

The results here generalise some of those in [A1-3] and should provide useful tools for extending other results such as the determination of the (ordered) K-theory and the noncommutative metrics of these algebras.

1. Preliminaries.

The deformations of $C(G/\Gamma)$ constructed in [LR2] are based on the following *standard assumptions:*

(S1) There is a compact abelian subgroup K of G and a homomorphism $\rho : \widehat{K} \mapsto G$ such that each $\rho(s)$ commutes with K,

(S2) Γ is a closed subgroup of G, each $x \in \Gamma$ commutes with K and satisfies

$$B_x(s) := x\rho(s)x^{-1}\rho(-s) \in K \textit{ for all } s \in \widehat{K} \textit{ and}$$
$$\langle B_x(s), t\rangle = \langle B_x(t), s\rangle \textit{ for all } s,t \in \widehat{K},$$

(S3) G/Γ is compact and $\Gamma \cap K = \{e\}$.

In [LR2, 4.8-11] it is then explained how one obtains a closed subgroup K_ρ of K by $K_\rho^\perp = \{t \in \widehat{K} \mid \rho(2t) \in K\Gamma\}$ and a homomorphism $\theta : \widehat{K} \mapsto K/K_\rho$. The subgroup G_ρ of G is then defined as the closure of

$$\{\rho(-2p)\theta(p)K_\rho\Gamma \mid p \in \widehat{K}\}$$

and it is shown that $K_\rho\Gamma$ is a closed normal subgroup of G_ρ with $G_\rho/K_\rho\Gamma$ a compact abelian group.

We define the partial Fourier transform by $\widehat{f}(x,s) := \int_K \overline{\langle k,s\rangle} f(xk)\,dk$ for $f \in C(G)$ as in [LR2] and

$$C_{b,1}(G) := \Big\{ f \in C_b(G) : \|f\|_{\infty,1} := \sum_s \sup_x |\widehat{f}(x,s)| < \infty \Big\}.$$

The space of functions we shall work with is $C_1(G/\Gamma) := C(G/\Gamma) \cap C_{b,1}(G)$. With the operations given by $f^*(x) = \overline{f(x)}$ and

$$f * g(x) = \sum_{s,t} \widehat{f}(x\rho(t), s)\widehat{g}(x\rho(-s), t) \tag{1.1}$$

we have a Banach *-algebra denoted $C_1(G/\Gamma, \rho)$. Its regular representation μ over $L^2(G)$ is described in [LR2, Section 1], and $C_r^*(\widehat{G}/\Gamma, \rho)$ is the C^*-closure of $\mu(C_1(G/\Gamma, \rho))$. Note that this definition is closely related to the Fell bundle approach in [AE1].

We refer to [LR2] for more details and other concepts not explained here, in fact this article is unreadable without [LR1-2].

2. The center of $C_r^*(\widehat{G}/\Gamma, \rho)$.

It was shown in [LR2, Theorem 4.15] that $C_1(G/G_\rho, \rho)$ with the product (1.1) is a dense subalgebra of the center $\mathcal{Z}C_r^*(\widehat{G}/\Gamma, \rho)$. However, this product will not be the pointwise product for functions in $C_1(G/G_\rho, \rho)$. We shall see that the center actually is isomorphic to $C(G/G_\rho^0)$ where G_ρ^0 is another subgroup of G. We thank the referee for discovering an error in our description, this means that Remark 4.16 in [LR2] is incorrect.

We shall also need some other subgroups of G, and a guiding example in this section is to take the Heisenberg manifolds as described in [LR2, Section 3] with μ and ν rational. All concepts in [LR2, Section 4] are used without further explanation. For the first new subgroup note that

$$\{\rho(-2p)\theta(p)K_\rho\Gamma \mid p \in \widehat{K}\} \supset \{\rho(-2p)\theta(p)K_\rho\Gamma \mid p \in K_\rho^\perp\} = \{\widetilde{\theta}(-p)\theta(p)K_\rho\Gamma \mid p \in K_\rho^\perp\}.$$

The following should then be obvious:

Lemma 2.1. *Take $K_0 = \{\widetilde{\theta}(-t)\theta(t)K_\rho \mid t \in K_\rho^\perp\}$. Then $K_\rho\Gamma \subset K_0\Gamma \subset G_\rho$ and*

$$K_0^\perp = \{s \in K_\rho^\perp \mid \langle\theta(s), t\rangle = \langle\theta(t), s\rangle \text{ for all } t \in K_\rho^\perp\}.$$

Furthermore, all $f \in C_1(G/G_\rho)$ satisfies $\widehat{f}(x,s) = 0$ for $s \notin K_0^\perp$.

In particular this means that we have $\langle\theta(s), t\rangle = \langle\theta(t), s\rangle$ for all $s,t \in K_0^\perp$, so there is a function $c : K_0^\perp \mapsto \mathbf{T}$ such that

$$\langle\theta(s), t\rangle = \frac{c(s)c(t)}{c(s+t)} \quad \text{for all} \quad s,t \in K_0^\perp. \tag{2.1}$$

Lemma 2.2. *There is a function $b : \widehat{K} \mapsto \mathbf{T}$ such that*

$$\frac{b(s)c(t)}{b(s+t)} = \langle \theta(s), t \rangle \quad \text{for all} \quad s \in \widehat{K},\ t \in K_0^{\perp}. \tag{2.2}$$

Proof. Pick one s_i from each equivalence class in $\widehat{K}/K_0^{\perp}$ and define b by

$$b(s_i + t) = c(t)\langle \theta(s_i), -t \rangle \text{ for } t \in K_0^{\perp}.$$

It is then straightforward to check that (2.2) holds.

Lemma 2.3. *For $f \in C_1(G/G_\rho)$ define*

$$\Phi(f)(x) = \sum_{s \in K_0^{\perp}} \widehat{f}(x\rho(s), s)c(s).$$

Then Φ is a 1-1 *-homomorphism from $C_1(G/G_\rho, \rho)$ with the product (1.1) into $C(G)$ equipped with the usual pointwise operations.

Proof. For $f, g \in C_1(G/G_\rho)$ we have by (1.1) and the definition of G_ρ that

$$\begin{aligned}
\Phi(f * g)(x) &= \sum_{s \in K_0^{\perp}} (f * g)\widehat{\ }(x\rho(s), s)c(s) \\
&= \sum_{s,t \in K_0^{\perp}} \widehat{f}(x\rho(2s - t), t)\widehat{g}(x\rho(s - t), s - t)c(s) \\
&= \sum \widehat{f}(x\rho(2s + t), t)\widehat{g}(x\rho(s), s)c(s + t) \\
&= \sum \widehat{f}(x\rho(t)\theta(s), t)\widehat{g}(x\rho(s), s)c(s + t) \\
&= \sum \widehat{f}(x\rho(t), t)\widehat{g}(x\rho(s), s)\langle \theta(s), t \rangle c(s + t) \\
&= \sum \widehat{f}(x\rho(t), t)\widehat{g}(x\rho(s), s)c(s)c(t) \\
&= \Phi(f)(x)\Phi(g)(x).
\end{aligned}$$

The *-operation is complex conjugation in both algebras, so

$$\begin{aligned}
\Phi(f^*)(x) &= \sum_{s \in K_0^{\perp}} \overline{\widehat{f}(x\rho(s), -s)}c(s) = \sum \overline{\widehat{f}(x\rho(-s), s)}c(-s) \\
&= \sum \overline{\widehat{f}(x\rho(s)\theta(-s), s)}c(-s) = \sum \overline{\widehat{f}(x\rho(s), s)\langle \theta(-s), s \rangle}c(-s) \\
&= \sum \overline{\widehat{f}(x\rho(s), s)c(s)} = \overline{\Phi(f)(x)}. \quad \square
\end{aligned}$$

In order to define the subgroup G_ρ^0 we need the following construction:

Lemma 2.4. *There is a continuous homomorphism $B : G_\rho \mapsto K/K_\rho$ satisfying $B(x)^2 = e$ and*

$$\langle B(x), s\rangle = \langle x\rho(s)x^{-1}\rho(-s), s\rangle = \pm 1 \quad \textit{for} \quad s \in K_\rho^\perp. \tag{2.3}$$

Proof. If $x \in K$ or $x = \rho(t)$ then $x\rho(s)x^{-1}\rho(-s) = e$, so we have $B(x) = e$. It follows from (S1-S3) that for $x, y \in \Gamma$ and $s, t \in \widehat{K}$ that

$$B_x(s+t) = B_x(s)B_x(t) \tag{2.4}$$

$$B_{xy}(s) = B_x(s)B_y(s). \tag{2.5}$$

For $x \in \Gamma$ and $s \in K_\rho^\perp$ we proved in [LR2, Lemma 4.11] that $B_x(2s) = e$. So if we look at the function $s \mapsto \langle B_x(s), s\rangle$ we have for $s, t \in K_\rho^\perp$ that

$$\begin{aligned}\langle B_x(s+t), s+t\rangle &= \langle B_x(s), s\rangle\langle B_x(s), t\rangle\langle B_x(t), s\rangle\langle B_x(t), t\rangle \\ &= \langle B_x(s), s\rangle\langle B_x(2s), t\rangle\langle B_x(t), t\rangle \\ &= \langle B_x(s), s\rangle\langle B_x(t), t\rangle.\end{aligned}$$

So there is an element $B(x) \in K/K_\rho$ such that

$$\langle B(x), s\rangle = \langle B_x(s), s\rangle = \langle x\rho(s)x^{-1}\rho(-s), s\rangle \quad \text{for} \quad s \in K_\rho^\perp.$$

This holds for all $x \in G_\rho$, and for a fixed $s \in K_\rho^\perp$ this expression is continuous in x. So (2.3-5) together with $B_x(2s) = e$ implies that B is a continuous homomorphism with $\langle B(x), s\rangle = \pm 1$ and therefore $B(x)^2 = e$ in K/K_ρ. □

Note the map B can be defined the same way on the group G_1 defined in [LR2, Lemma 4.7], but this is not needed here.

Lemma 2.5. *Define*

$$G_\rho^0 := \{yB(y^{-1})K_\rho \mid y \in G_\rho\} = \{\rho(-2p)\theta(p)zB(z^{-1})K_\rho \mid p \in \widehat{K}, z \in \Gamma\}^-.$$

Then the homomorphism Φ in Lemma 2.3 has dense image in $C(G/G_\rho^0)$.

Proof. If $f \in C_1(G/G_\rho)$, then $\widehat{f}(xy, s) = \widehat{f}(x, s)$ for $y \in G_\rho$. So from the definition of Φ we have

$$\begin{aligned}\Phi(f)(xy) &= \sum_{s\in K_0^\perp} \widehat{f}(xy\rho(s), s)c(s) = \sum \widehat{f}(x\rho(s)y, s)\langle B_y(s), s\rangle c(s) \\ &= \sum \widehat{f}(x\rho(s), s)\langle B(y), s\rangle c(s) = \sum \widehat{f}(xB(y)\rho(s), s)c(s) \\ &= \Phi(f)(xB(y)).\end{aligned}$$

Thus $\Phi(f)(xyB(y)^{-1}) = \Phi(f)(x)$. Φ is a bijection between $C_1(G/G_\rho)$ and $C_1(G/G_\rho^0)$, so the image is dense. $\square$

We want to show that Φ extends to an isomorphism between $\mathcal{Z}C_r^*(\widehat{G}/\Gamma,\rho)$ and $C(G/G_\rho^0)$, and since $C_r^*(\widehat{G}/\Gamma,\rho)$ is defined by using the regular representation μ this will follow from:

Proposition 2.6. *For $f \in C_1(G/G_\rho)$ the unitary operator $U = \sum_{s\in\widehat{K}} b(s)L_{\rho(s)}P_s$ satisfies*

$$U\mu(f)U^* = M(\Phi(f)).$$

Proof. From Propositions 1.3 and (1.11) in [LR2] and Lemma 2.2 above we have

$$\begin{aligned}
U\mu(f)U^* &= \sum_{s,t\in\widehat{K}} b(t)L_{\rho(t)}P_t\mu(f)\overline{b(s)}L_{\rho(-s)}P_s \\
&= \sum b(t)L_{\rho(s+t)}P_tM(f)\overline{b(s)}L_{\rho(-s-t)}P_s \\
&= \sum b(t)L_{\rho(s+t)}M(\widehat{f}(\cdot,t-s))\overline{b(s)}L_{\rho(-s-t)}P_s \\
&= \sum b(t+s)\overline{b(s)}L_{\rho(2s+t)}M(\widehat{f}(\cdot,t))L_{\rho(-2s-t)}P_s \\
&= \sum_{s\in\widehat{K},t\in K_0^\perp} b(t+s)\overline{b(s)}M(\widehat{f}(\cdot\,\rho(2s+t),t))P_s \\
&= \sum b(t+s)\overline{b(s)}\langle\theta(s),t\rangle M(\widehat{f}(\cdot\,\rho(t),t))P_s \\
&= \sum c(t)M(\widehat{f}(\cdot\,\rho(t),t))P_s = M(\Phi(f)). \quad \square
\end{aligned}$$

Note that every $x \in G_\rho$ satisfies (S2), so by [LR2, Theorem 4.3] β_x defined by $\beta_x(f)(y) = f(yx)$ extends to a *-automorphism of $C_r^*(\widehat{G}/\Gamma,\rho)$. If we also look at part (1) and (2) of the proof of [LR2, Theorem 4.15], we see that it can be rephrased as

$$\mathcal{Z}C_r^*(\widehat{G}/\Gamma,\rho) = \{a \in C_r^*(\widehat{G}/\Gamma,\rho) \mid \beta_x(a) = a \text{ for all } x \in G_\rho\}. \tag{2.6}$$

Theorem 2.7. *The map Φ extends to a C^*-isomorphism between $\mathcal{Z}C_r^*(\widehat{G}/\Gamma,\rho)$ and $C(G/G_\rho^0)$ with the pointwise product.*

Proof. This now follows, just note that in part (2) of the proof of [LR2, Theorem 4.15] it was shown that $\mu(C_1(G/G_\rho))$ is dense in the center of $C_r^*(\widehat{G}/\Gamma,\rho)$.

Remarks 2.8. The map $y \mapsto yB(y)^{-1}$ is an isomorphism between the groups G_ρ/K_ρ and G_ρ^0/K_ρ. However, this will not imply that the groups G_ρ and G_ρ^0 themselves are isomorphic or that G/G_ρ is homeomorphic to G/G_ρ^0. Also note that $G = G_\rho \iff G = G_\rho^0$ which again by [LR2, Theorem 4.15] is equivalent to $C_r^*(\widehat{G}/\Gamma,\rho)$ being simple. We shall see in Section 5 that B can be nontrivial and $G_\rho \neq G_\rho^0$.

3. The conditional expectation onto the center and traces on $C_r^*(\widehat{G}/\Gamma,\rho)$.

We have now proved that $\mathcal{Z}C_r^*(\widehat{G}/\Gamma,\rho)$ is generated by $\{\mu(f) \mid f \in C_1(G/G_\rho)\}$ and is via the map Φ isomorphic to $C(G/G_\rho^0)$. From (2.6) we get the natural conditional expectation E of $C_r^*(\widehat{G}/\Gamma,\rho)$ onto its center by

$$E(a) = \int_{G_\rho/\Gamma} \beta_x(a)\,dx.$$

Note that G_ρ/Γ is not necessarily a group, but as noted in the preliminaries $K_\rho\Gamma$ is a closed normal subgroup of G_ρ with $G_\rho/K_\rho\Gamma$ a compact abelian group. (The same is true if we replace G_ρ with G_ρ^0.) This means that the map E is given by

$$E(a) = \int_{G_\rho/\Gamma} \beta_x(a)\,dx = \int_{G_\rho/K_\rho\Gamma}\int_{K_\rho} \beta_{yk}(a)\,dkdy. \tag{3.1}$$

Lemma 3.1. *$E^0(a) = \Phi(E(a))$ defines a conditional expectation from $C_r^*(\widehat{G}/\Gamma,\rho)$ onto $C(G/G_\rho^0)$. For $f \in C_1(G/\Gamma,\rho)$ we have*

$$E^0(\mu(f))(x) = \sum_{s\in K_0^\perp} c(s)\int_{G_\rho^0/K_\rho\Gamma} \widehat{f}(xz\rho(s),s)\,dz. \tag{3.2}$$

Proof. With $\widetilde{E}(f)(x) = \int_{G_\rho/\Gamma} f(xz)\,dz = \int_{G_\rho/K_\rho\Gamma}\int_{K_\rho} f(xyk)\,dkdy$ we have $E(\mu(f)) = \mu(\widetilde{E}(f))$, so

$$\begin{aligned} E^0(\mu(f))(x) &= \Phi(\widetilde{E}(f))(x) \\ &= \sum_{s\in K_0^\perp} c(s)\widetilde{E}(f)\widehat{\ }(x\rho(s),s) \\ &= \sum c(s)\int_{G_\rho/\Gamma} \widehat{f}(x\rho(s)z,s)\,dz. \end{aligned}$$

Since $s \in K_0^\perp \subset K_\rho^\perp$ we have

$$\begin{aligned} E^0(\mu(f))(x) &= \sum c(s)\int_{G_\rho/K_\rho\Gamma} \widehat{f}(x\rho(s)y,s)\,dy \\ &= \sum c(s)\int \widehat{f}(xy\rho(s)B_{y^{-1}}(s),s)\,dy \\ &= \sum c(s)\int \widehat{f}(xy\rho(s),s)\langle B_{y^{-1}}(s),s\rangle\,dy. \end{aligned}$$

From Lemma 2.4 we have $\langle B_{y^{-1}}(s),s\rangle = \langle B(y^{-1}),s\rangle$, so

$$\begin{aligned} E^0(\mu(f))(x) &= \sum c(s)\int_{G_\rho/K_\rho\Gamma} \widehat{f}(xyB(y^{-1})\rho(s),s)\,dy \\ &= \sum c(s)\int_{G_\rho^0/K_\rho\Gamma} \widehat{f}(xz\rho(s),s)\,dz. \end{aligned}$$

Lemma 3.2. *For $f, g \in C_1(G/\Gamma, \rho)$ we have $E^0(\mu(f * g)) = E^0(\mu(g * f))$.*

Proof. There are no problems in interchanging integrals and sums here, so for $s \in K_0^\perp$

$$\int_{G_\rho^0/K_\rho\Gamma} (f * g)\widehat{\ }(xz\rho(s), s)\, dz = \int \sum_{t \in \widehat{K}} \widehat{f}(xz\rho(2s-t), t)\widehat{g}(xz\rho(s-t), s-t)\, dz$$
$$= \sum \int \widehat{f}(xz\rho(s+t), s-t)\widehat{g}(xz\rho(t), t)\, dz.$$

Here we used the substitution $t \mapsto s - t$, we continue with the substitution $zK_\rho \mapsto z\rho(-2t)\theta(t)K_\rho$ to get

$$\int (f * g)\widehat{\ }(xz\rho(s), s)\, dz = \sum \int \widehat{f}(xz\rho(s-t), s-t)\widehat{g}(xz\rho(-t), t)\langle\theta(t), s-t+t\rangle\, dz.$$

From (4.9) in [LR2] we have that $\widetilde{\theta}(s)\Gamma = \rho(2s)\Gamma$ for $s \in K_\rho^\perp$, which together with $\langle\theta(t), s\rangle = \langle\widetilde{\theta}(s), t\rangle$ gives

$$\int (f * g)\widehat{\ }(xz\rho(s), s)\, dz = \sum \int \widehat{f}(xz\rho(s-t), s-t)\widehat{g}(xz\rho(2s-t), t)\, dz$$
$$= \int (g * f)\widehat{\ }(xz\rho(s), s)\, dz.$$

This holds for all $s \in K_0^\perp$, so from (3.2) we have $E^0(\mu(f * g)) = E^0(\mu(g * f))$. $\square$

We can now prove the following generalisation of [A3, Corollary 3.11]:

Theorem 3.3. *The conditional expectation from $C_r^*(\widehat{G}/\Gamma, \rho)$ onto $C(G/G_\rho^0)$ given by $E^0 = \Phi \circ E$ satisfies $\alpha_x \circ E^0 = E^0 \circ \alpha_x$ and $E^0(ab) = E^0(ba)$. There is a 1-1 correspondence between normalised traces τ on $C_r^*(\widehat{G}/\Gamma, \rho)$ and probability measures ν on G/G_ρ^0 given by $\tau = \nu \circ E^0$. τ is faithful if and only if ν is.*

Proof. The α-invariance is obvious. Since $\mu(C_1(G/\Gamma, \rho))$ is dense in $C_r^*(\widehat{G}/\Gamma, \rho)$, the first part follows from Lemma 3.2. Hence $\tau = \nu \circ E^0$ is a normalised trace for all probability measures ν on G/G_ρ^0. Since E^0 is faithful, it is immediate that τ is faithful if and only if ν is.

Conversely, if τ is a normalised trace on $C_r^*(\widehat{G}/\Gamma, \rho)$, it follows from [LR2, Lemma 4.12] that $\tau \circ \beta_k = \tau$ for $k \in K_\rho$. So $\tau(a) = \tau(b)$ where $b = \int_{K_\rho} \beta_k(a)\, dk$. From [LR2, Lemma 4.14] it follows that $\tau \circ \beta_y(b) = \tau(b)$ for $y \in G_\rho$, so

$$\tau(a) = \int_{G_\rho/K_\rho\Gamma} \int_{K_\rho} \tau \circ \beta_{yk}(a)\, dk dy = \tau(E(a)).$$

Let ν be the measure on G/G_ρ^0 given by $\nu(f) = \tau(\Phi^{-1}(f))$ for $f \in C(G/G_\rho^0)$, then $\tau(a) = \tau(E(a)) = \nu(\Phi \circ E(a)) = \nu(E^0(a))$. $\square$

From [LR2, Theorem 4.15] we now have

Corollary 3.4. *If $G = G_\rho$ (which is equivalent to $G = G_\rho^0$), there is a unique trace on the simple C^*-algebra $C_r^*(\widehat{G}/\Gamma, \rho)$.*

4. Quasi-invariant measures on G/Γ and representations over $L^2(G/\Gamma)$.

In this section we shall look at traces on $C_r^*(\widehat{G}/\Gamma, \rho)$ obtained from a G-quasi-invariant measure ν on G/G_ρ^0. We shall see that the corresponding GNS-representation can be realised over $L^2(G/\Gamma, \nu)$.

If ν is a G-quasi-invariant measure on G/G_ρ^0, there is a function $h \in C(G \times G/G_\rho^0)$ such that $\nu(\alpha_x(f)) = \nu(h(x, \cdot)f)$ for $f \in C(G/G_\rho^0)$, in fact $h(x, y) = \frac{r(xy)}{r(y)}$ where r is a continuous rho-function on G corresponding to ν. If we take $z_x = \Phi^{-1}[h(x, \cdot)]$ and use Theorem 3.3 on such measures, we get

Corollary 4.1. *If ν is a G-quasi-invariant probability measure on G/G_ρ^0 and $\tau = \nu \circ E^0$, there is a continuous function $x \in G \mapsto z_x \in \mathcal{Z}C_r^*(\widehat{G}/\Gamma, \rho)^+$ such that $\tau \circ \alpha_x(a) = \tau(z_x a)$ for all $a \in C_r^*(\widehat{G}/\Gamma, \rho)$.*

Since G_ρ^0/Γ has a G_ρ^0-invariant probability measure and G/Γ is compact, it follows that G/Γ has a G-invariant probability measure if and only if G/G_ρ^0 has one. And if so, it is unique. However, note that the compactness of G/Γ does *not* imply the existence of a G-invariant probability measure.

Corollary 4.2. *If G/Γ has a G-invariant probability measure, then there is a unique normalised α_G-invariant trace on $C_r^*(\widehat{G}/\Gamma, \rho)$.*

The regular representation of $C_r^*(\widehat{G}/\Gamma, \rho)$ is over $L^2(G)$, but it seems natural to also represent it over $L^2(G/\Gamma)$. This is obtained by using the GNS-representation obtained from τ as in Corollary 4.1.

Lemma 4.3. *G/Γ has a G-quasi-invariant probability measure ν such that $\nu(g^* * f) = \nu(\overline{g}f)$ for $f, g \in C_1(G/\Gamma)$.*

Proof. First we shall need the closed subgroup $G_2 = \{\rho(t)K\Gamma \mid t \in \widehat{K}\}^-$ of G. Exactly as in [LR2, Lemma 4.7] one proves that $K\Gamma$ is a closed, normal subgroup of G_2, and $G_2/K\Gamma$ is a compact abelian group. If we now take a G-quasi-invariant probability measure on G/G_2, Haar-measures on $G_2/K\Gamma$ and K, then

$$\int f \, d\nu = \int_{G/G_2} \int_{G_2/K\Gamma} \int_K f(xyk) \, dk \, dy \, dx$$

defines a G-quasi-invariant probability measure on G/Γ. So for $f, g \in C_1(G/\Gamma)$ we have

$$\begin{aligned}
\int_K g^* * f(xk)\, dk &= \int_K \sum_{s,t} \overline{\widehat{g}(xk\rho(t), -s)}\widehat{f}(xk\rho(-s), t)\, dk \\
&= \sum_t \overline{\widehat{g}(x\rho(t), t)}\widehat{f}(x\rho(t), t). \\
\nu(g^* * f) &= \int_{G/G_2}\int_{G_2/K\Gamma} \sum_t \overline{\widehat{g}(xy\rho(t), t)}\widehat{f}(xy\rho(t), t)\, dy\, dx \quad (y \mapsto y\rho(-t)) \\
&= \int_{G/G_2}\int_{G_2/K\Gamma} \sum_t \overline{\widehat{g}(xy, t)}\widehat{f}(xy, t)\, dy\, dx \\
&= \int_{G/G_2}\int_{G_2/K\Gamma} \int_K \overline{g(xyk)} f(xyk)\, dk\, dy\, dx \\
&= \nu(\overline{g} f).
\end{aligned}$$

Lemma 4.4. *Let ν be a G-quasi-invariant probability measure on G/Γ. Then there is a function $\phi \in C_c(G)$ such that for all $f \in C_1(G/\Gamma)$*

$$\langle \mu(f)\phi \,|\, \phi\rangle = \int_{G/\Gamma} f\, d\nu. \tag{4.1}$$

Proof. Let r be a continuous rho-function as in [FD, Section III.13.2] or [LR1, Section 2], we may assume that $r(xk) = r(x)$ for $k \in K$. So for $f \in C_c(G)$

$$\int_{G/\Gamma}\int_\Gamma f(xh)\, dh\, d\nu(x) = \int_G f(x)r(x)\, dx.$$

Since G/Γ is compact, there is a function $\phi_0 \in C_c(G)$ such that

$$\int_\Gamma \phi_0(xh)\, dh = 1 \quad \text{for all} \quad x \in G. \tag{4.2}$$

K is compact and commutes with Γ, so we may assume that $\phi_0(xk) = \phi_0(x)$ for $k \in K$. Now take $\phi(x) = [\phi_0(x^{-1})r(x^{-1})\Delta(x^{-1})]^{\frac{1}{2}}$. Since $P_0\phi = \phi$, we get from [LR2, Proposition 1.3] that

$$\begin{aligned}
\langle \mu(f)\phi \,|\, \phi\rangle = \langle M(f)\phi \,|\, \phi\rangle &= \int_G f(x^{-1})\phi(x)^2\, dx \\
&= \int_G f(x^{-1})\phi_0(x^{-1})r(x^{-1})\Delta(x^{-1})\, dx = \int_G f(x)\phi_0(x)r(x)\, dx \\
&= \int_{G/\Gamma}\int_\Gamma f(y)\phi_0(yh)\, dh\, d\nu(y) = \int_{G/\Gamma} f(y)\, d\nu(y). \quad \square
\end{aligned}$$

These two lemmas show that $C_r^*(\widehat{G}/\Gamma,\rho)$ – which was defined over $L^2(G)$ – also can be represented over $L^2(G/\Gamma,\nu)$. This is done as follows: Define $V : C_1(G/\Gamma)\mapsto L^2(G)$ by $Vf=\mu(f)\phi$. Then $\langle Vf\,|\,Vg\rangle=\langle\mu(g^* * f)\phi\,|\,\phi\rangle=\nu(g^* * f)=\nu(\overline{g}f)$, so V extends to a partial isometry from $L^2(G/\Gamma,\nu)$ into $L^2(G)$ with $V^*V=I$. We also have

$$V^*\mu(f)Vg=V^*\mu(f)\mu(g)\phi=V^*\mu(f*g)\phi=V^*Vf*g=f*g.$$

Thus the GNS-representation of $C_r^*(\widehat{G}/\Gamma,\rho)$ corresponding to τ is really over $L^2(G/\Gamma,\nu)$ and V sets up an equivalence with a sub-representation of the regular representation μ. It is faithful, since $\langle V^*aV1\,|\,1\rangle=\langle a\phi\,|\,\phi\rangle=\tau(a)$ and τ is faithful. Thus we have shown:

Theorem 4.5. *If ν is a G-quasi-invariant probability measure on G/Γ as in Lemma 4.3, then the GNS-representation of $C_r^*(\widehat{G}/\Gamma,\rho)$ corresponding to ν is a faithful representation over $L^2(G/\Gamma,\nu)$ and equivalent to a sub-representation of the regular representation μ.*

5. The Heisenberg manifolds revisited.

Let us go back to the Heisenberg manifolds in [R2], we shall use our description in [LR2, Section 3]; so $C_r^*(\widehat{G}/\Gamma,\rho)\cong D_{\mu,\nu}$ in the terminology of [LR2] and [A1-3], we only look at the case with $c=1$. We only state the results and leave the computations to the reader.

If $\mu,\nu\in\mathbf{Q}$ one finds that $K_\rho^\perp=q\mathbf{Z}$, where q is the smallest integer $\neq 0$ with both $2\mu q$ and $2\nu q\in\mathbf{Z}$. $\widetilde{\theta}(t)=\theta(t)=(-1)^{4q\mu\nu t}$ for $t\in q\mathbf{Z}$, so $K_\rho^\perp=K_0^\perp$. For any $g=(x,y,z)\in G$ define $B(g)=e^{2\pi i q(\mu y-\nu x)}$ and $B(g)$ satisfies (2.3) for $g\in\Gamma$. If e.g. $\mu=\nu=\frac{1}{2}$, then $q=1$ and $K_\rho=\{e\}$, but B is not trivial on Γ, so $G_\rho\neq G_\rho^0$. However, since B is defined on the whole group G, we have $G_\rho\cong G_\rho^0$ and G/G_ρ is homeomorphic to G/G_ρ^0.

On the other hand if μ or ν is irrational, then $K_\rho=K_0=K$, so $G_\rho=G_\rho^0=\{\rho(2n)K\Gamma\mid n\in\mathbf{Z}\}^-$ and is a normal subgroup of G with $G/G_\rho\cong\mathbf{T}^2/T_0$ where T_0 is the closed subgroup generated by $(\exp 4\pi i\mu,\exp 4\pi i\nu)$. So Theorem 3.3 is our version of [A3, Corollary 3.11].

Many of the structural results about $D_{\mu,\nu}$ in [A1-3] can be proved using the present description. Let us briefly illustrate this by showing the existence of projections á la Rieffel using only functions in $C_1(G/\Gamma)$. Let

$$\begin{aligned} f(x,y,z)&=F_\mu(x) \qquad & g(x,y,z)&=G_\mu(x)z\exp(-2\pi i[x]y)\\ h(x,y,z)&=F_\nu(y) \qquad & k(x,y,z)&=G_\nu(y)z\exp(-2\pi i x([y]-y)) \end{aligned}$$

where F_μ and G_μ are continuous functions satisfying a slightly modified version of [R1, Theorem 1.1 (1-3)]. In particular we need $G_\mu(0)=0$ in order to make g and k continuous.

Computations as in [LR2, Proposition 3.6] show that $p = f + g + g^*$ and $q = h + k + k^*$ are projections in $D_{\mu,\nu}$ with $\tau(p) = 2\mu$ and $\tau(q) = 2\nu$ for any normalised trace τ.

This is used in [A1, Theorem 1] to show that $\mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z} \subset K_0(D_{\mu,\nu})$. To show equality Abadie proves in [A3] that $D_{\mu,\nu}$ can be embedded in an AF-algebra. This can also be done using only functions in $C_1(G/\Gamma)$. The function

$$w(x, y, z) = z \exp(-2\pi i[x]y)$$

is in $L^\infty(G/\Gamma)$, but is not continuous. Since $w = \widehat{w}(\cdot, 1)$, the regular representation in [LR2, Proposition 1.3] can still be used on w to get a unitary operator W. The C^*-algebra $\mathcal{B}$ generated by $D_{\mu,\nu}$ and W is invariant under the automorphisms β_t and we have $\beta_t(W) = tW$ for all $t \in \mathbf{T}$. It is then standard to show that $\mathcal{B}$ is in fact the crossed product of a norm-closed subalgebra of $L^\infty(\mathbf{T}^2)$ with $\mathbf{Z}$ as in [A3, Theorem 2.3], and by classical results by Pimsner in [P] it follows that $\mathcal{B}$ can be embedded in an AF-algebra. Abadie uses this to determine the ordered K-theory of $D_{\mu,\nu}$ and to describe when two such algebras are isomorphic: In most cases $D_{\mu,\nu} \cong D_{\mu',\nu'}$ if and only if (μ, ν) and (μ', ν') belong to the same orbit under the natural action of GL(2,$\mathbf{Z}$) on $\mathbf{T}^2$, see [AE2, Theorem 2.2] and [A3, Corollary 3.17].

Note that the functions above (except w) can be taken to be C^∞-functions, so also the cyclic cohomology of $D_{\mu,\nu}$ can be studied. It should then be possible to extend the results for the Heisenberg manifolds to the more general algebras $C^*_r(\widehat{G}/\Gamma, \rho)$ described here by finding functions in $C_1(G/\Gamma, \rho)$ having the right properties. It is our belief that the presentation of these algebras given in [LR1-2] and here will be useful for such constructions. The noncommutative metrics studied by Rieffel and Weaver in [R3] and [W] are examples of this.

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